

# SPECIAL REPRESENTATIONS OF WEYL GROUPS: A POSITIVITY PROPERTY

G. LUSZTIG

## INTRODUCTION

Let  $W$  be an irreducible Weyl group with length function  $l : W \rightarrow \mathbf{N}$  and let  $S = \{s \in W; l(s) = 1\}$ . Let  $\text{Irr}W$  be a set of representatives for the isomorphism classes of irreducible representations of  $W$  (over  $\mathbf{C}$ ). In [L1] a certain subset of  $\text{Irr}W$  was defined. The representations in this subset were later called *special representations*; they play a key role in the classification of unipotent representations of a reductive group over a finite field  $\mathbf{F}_q$  for which  $W$  is the Weyl group. (The definition of special representations is reviewed in 3.1.)

It will be convenient to replace irreducible representations of  $W$  with the corresponding simple modules of the asymptotic Hecke algebra  $\mathbf{J}$  (see [L6, 18.3]) associated to  $W$  via the canonical isomorphism  $\psi : \mathbf{C}[W] \xrightarrow{\sim} \mathbf{J}$  (see 3.1); let  $E_\infty$  be the simple  $\mathbf{J}$ -module corresponding to  $E \in \text{Irr}W$  under  $\psi$ .

In this paper we show that a special representation  $E$  of  $W$  is characterized by the following positivity property of  $E_\infty$ : there exists a  $\mathbf{C}$ -basis of  $E_\infty$  such that any element  $t_u$  in the standard basis of  $\mathbf{J}$  acts in this basis through a matrix with all entries in  $\mathbf{R}_{\geq 0}$ .

The fact that for a special representation  $E$ ,  $E_\infty$  has the positivity property above was pointed out (in the case where  $W$  is of classical type) in [L9]. In this paper I will recall the argument of [L9] (see 3.3) and I give two other proofs which apply for any  $W$ . One of these proofs (see 4.4) is based on the interpretation [L3], [BFO], of  $\mathbf{J}$  (or its part attached to a fixed two-sided cell) in terms of  $G$ -equivariant vector bundles on  $X \times X$  where  $X$  is a finite set with an action of a finite group  $G$ . Another proof (see Section 2) is based on the use of Perron's theorem for matrices with all entries in  $\mathbf{R}_{>0}$ . (Previously, Perron's theorem has been used in the context of canonical bases in quantum groups in the study [L5] of total positivity and, very recently, in the context of the canonical basis [KL1] of  $\mathbf{C}[W]$ , in [KM]; in both cases the positivity properties of the appropriate canonical bases were used). We also show that the Hecke algebra representation corresponding to a special representation  $E$  can be realized essentially by a  $W$ -graph (in the sense

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of [KL1]) in which all labels are natural numbers. Some of our results admit also an extension to the case of affine Weyl groups (see Section 5).

### 1. STATEMENT OF THE MAIN THEOREM

**1.1.** Let  $v$  be an indeterminate and let  $\mathcal{A} = \mathbf{Z}[v, v^{-1}]$ . Let  $\mathcal{H}$  be the Hecke algebra of  $W$  that is, the associative  $\mathcal{A}$ -algebra with 1 with an  $\mathcal{A}$ -basis  $\{T_w; w \in W\}$  (where  $T_1 = 1$ ) and with multiplication such that  $T_w T_{w'} = T_{ww'}$  if  $l(ww') = l(w) + l(w')$  and  $(T_s + 1)(T_s - v^2) = 0$  if  $s \in S$ . Let  $\{c_w; w \in W\}$  be the  $\mathcal{A}$ -basis of  $\mathcal{H}$  denoted by  $\{C'_w; w \in W\}$  in [KL1] (with  $q = v^2$ ); see also [L6, 5.2]. For example, if  $s \in S$ , we have  $c_s = v^{-1}T_s + v^{-1}$ . The left cells and two-sided cells of  $W$  are the equivalence classes for the relations  $\sim_L$  and  $\sim_{LR}$  on  $W$  defined in [KL1], see also [L6, 8.1]; we shall write  $\sim$  instead of  $\sim_L$ . For  $x, y$  in  $W$  we have  $c_x c_y = \sum_{z \in W} h_{x,y,z} c_z$  where  $h_{x,y,z} \in \mathbf{N}[v, v^{-1}]$ . As in [L6, 13.6], for  $z \in W$  we define  $a(z) \in \mathbf{N}$  by  $h_{x,y,z} \in v^{a(z)} \mathbf{Z}[v^{-1}]$  for all  $x, y$  in  $W$  and  $h_{x,y,z} \notin v^{a(z)-1} \mathbf{Z}[v^{-1}]$  for some  $x, y$  in  $W$ . (For example,  $a(1) = 0$  and  $a(s) = 1$  if  $s \in S$ .) For  $x, y, z$  in  $W$  we have  $h_{x,y,z} = \gamma_{x,y,z^{-1}} v^{a(z)} \pmod{v^{a(z)-1} \mathbf{Z}[v^{-1}]}$  where  $\gamma_{x,y,z^{-1}} \in \mathbf{N}$  is well defined. Let  $\mathbf{J}$  be the  $\mathbf{C}$ -vector space with basis  $\{t_w; w \in W\}$ . For  $x, y$  in  $W$  we set  $t_x t_y = \sum_{z \in W} \gamma_{x,y,z^{-1}} t_z \in \mathbf{J}$ . This defines a structure of associative  $\mathbf{C}$ -algebra on  $\mathbf{J}$  with unit element of the form  $\sum_{d \in \mathcal{D}} t_d$  where  $\mathcal{D}$  is a certain subset of the set of involutions in  $W$ , see [L6, 18.3]. For any subset  $X$  of  $W$  let  $\mathbf{J}_X$  be the subspace of  $\mathbf{J}$  with basis  $\{t_w; w \in X\}$ ; let  $\mathbf{J}_X^+$  be the set of elements of the form  $\sum_{w \in X} f_w t_w \in \mathbf{J}_X$  with  $f_w \in \mathbf{R}_{>0}$  for all  $w \in X$ . We have  $\mathbf{J} = \bigoplus_{\mathbf{c}} \mathbf{J}_{\mathbf{c}}$  where  $\mathbf{c}$  runs over the two-sided cells of  $W$ . Each  $\mathbf{J}_{\mathbf{c}}$  is a subalgebra of  $\mathbf{J}$  with unit element  $\sum_{d \in \mathcal{D}_{\mathbf{c}}} t_d$  where  $\mathcal{D}_{\mathbf{c}} = \mathbf{c} \cap \mathcal{D}$ ; moreover,  $\mathbf{J}_{\mathbf{c}} \mathbf{J}_{\mathbf{c}'} = 0$  if  $\mathbf{c} \neq \mathbf{c}'$ .

*Until the end of Section 4 we fix a two-sided cell  $\mathbf{c}$ .*

Let  $L$  be the set of left cells that are contained in  $\mathbf{c}$ . We have

$$\mathbf{c} = \sqcup_{\Gamma \in L} \Gamma = \sqcup_{\Gamma, \Gamma' \text{ in } L} (\Gamma \cap \Gamma'^{-1});$$

moreover,  $\Gamma \cap \Gamma'^{-1} \neq \emptyset$  for any  $\Gamma, \Gamma'$  in  $L$ . It follows that

$$\mathbf{J}_{\mathbf{c}} = \bigoplus_{\Gamma \in L} \mathbf{J}_{\Gamma} = \bigoplus_{\Gamma, \Gamma' \text{ in } L} \mathbf{J}_{\Gamma \cap \Gamma'^{-1}};$$

moreover,  $\mathbf{J}_{\Gamma \cap \Gamma'^{-1}} \neq 0$ . Note that for  $\Gamma \in L$ ,  $\mathbf{J}_{\Gamma}$  is a left ideal of  $\mathbf{J}_{\mathbf{c}}$ .

A line  $\mathcal{L}$  in  $\mathbf{J}_{\Gamma \cap \Gamma'^{-1}}$  is said to be *positive* if  $\mathcal{L}^+ := \mathcal{L} \cap \mathbf{J}_{\Gamma \cap \Gamma'^{-1}}^+ \neq \emptyset$ ; in this case,  $\mathcal{L}^+$  consists of all  $\mathbf{R}_{>0}$ -multiples of a single nonzero vector. We now state our main result.

**Theorem 1.2.** (a) *Let  $\Gamma \in L$ . There is a unique left ideal  $M_{\Gamma}$  of  $\mathbf{J}_{\mathbf{c}}$  such that property  $(\heartsuit)$  below holds:*

*$(\heartsuit)$   $M_{\Gamma} = \bigoplus_{\Gamma' \in L} M_{\Gamma, \Gamma'}$  where for any  $\Gamma' \in L$ ,  $M_{\Gamma, \Gamma'} := M_{\Gamma} \cap \mathbf{J}_{\Gamma \cap \Gamma'^{-1}}$  is a positive line.*

(b) *Let  $\Gamma \in L, \Gamma' \in L, u \in \mathbf{c}$ . We have  $u \in \tilde{\Gamma} \cap \tilde{\Gamma}'^{-1}$  for well-defined  $\tilde{\Gamma}, \tilde{\Gamma}'$  in  $L$ . If  $\tilde{\Gamma} \neq \tilde{\Gamma}'$ , then  $t_u M_{\Gamma, \Gamma'} = 0$ . If  $\tilde{\Gamma} = \tilde{\Gamma}'$ , then  $t_u M_{\Gamma, \Gamma'} = M_{\Gamma, \tilde{\Gamma}'}$  and  $t_u M_{\Gamma, \Gamma'}^+ = M_{\Gamma, \tilde{\Gamma}'}^+$ .*

(c) The  $\mathbf{J}_{\mathbf{c}}$ -module  $M_{\Gamma}$  in (a) is simple. Its isomorphism class is independent of  $\Gamma \in L$ .

(d) The subspace  $\mathbf{I} = \bigoplus_{\Gamma \in L} M_{\Gamma}$  of  $\mathbf{J}_{\mathbf{c}}$  is a simple two-sided ideal of  $\mathbf{J}_{\mathbf{c}}$ .

(e) Let  $\Gamma, \Gamma', \tilde{\Gamma}, \tilde{\Gamma}'$  be in  $L$ . If  $\Gamma \neq \tilde{\Gamma}'$  then  $M_{\Gamma, \Gamma'} M_{\tilde{\Gamma}, \tilde{\Gamma}'} = 0$ . If  $\Gamma = \tilde{\Gamma}'$ , then multiplication in  $\mathbf{J}_{\mathbf{c}}$  defines an isomorphism  $M_{\Gamma, \Gamma'} \otimes M_{\tilde{\Gamma}, \Gamma} \xrightarrow{\sim} M_{\tilde{\Gamma}, \Gamma'}$  and a surjective map  $M_{\Gamma, \Gamma'}^+ \times M_{\tilde{\Gamma}, \Gamma}^+ \rightarrow M_{\tilde{\Gamma}, \Gamma'}^+$ .

(f) Let  $\Gamma, \Gamma'$  be in  $L$ . The antiautomorphism  $\theta : \mathbf{J}_{\mathbf{c}} \rightarrow \mathbf{J}_{\mathbf{c}}$  given by  $t_x \mapsto t_{x^{-1}}$  for all  $x \in \mathbf{c}$  maps  $M_{\Gamma, \Gamma'}$  onto  $M_{\Gamma', \Gamma}$  and  $M_{\Gamma, \Gamma'}^+$  onto  $M_{\Gamma', \Gamma}^+$ .

The proof is given in Section 2.

**1.3.** As a consequence of Theorem 1.2, the simple  $\mathbf{J}_{\mathbf{c}}$ -module  $M_{\Gamma}$  admits a  $\mathbf{C}$ -basis  $\{\tilde{e}_{\Gamma'}; \Gamma' \in L\}$  with the following property:

(i) If  $u \in \mathbf{c}$  and  $\Gamma' \in L$ , then  $t_u \tilde{e}_{\Gamma'}$  is an  $\mathbf{R}_{\geq 0}$ -linear combination of elements  $\tilde{e}_{\Gamma''}$  with  $\Gamma'' \in L$ ; more precisely, if  $u \in \tilde{\Gamma} \cap \tilde{\Gamma}'^{-1}$  with  $\tilde{\Gamma}, \tilde{\Gamma}'$  in  $L$ , then

$$t_u \tilde{e}_{\Gamma'} = \lambda_{u, \Gamma', \tilde{\Gamma}} \tilde{e}_{\tilde{\Gamma}'},$$

with  $\lambda_{u, \Gamma', \tilde{\Gamma}} \in \mathbf{R}_{> 0}$  if  $\tilde{\Gamma} = \Gamma'$  and  $\lambda_{u, \Gamma', \tilde{\Gamma}} = 0$  if  $\tilde{\Gamma} \neq \Gamma'$ .

Indeed, we can take for  $\tilde{e}_{\Gamma'}$  any element of  $M_{\Gamma, \Gamma'}^+$  and we use 1.2(b).

**1.4.** Let  $\leq$  be the standard partial order on  $W$ . By [KL1], to any  $y \neq w$  in  $W$  one can attach a number  $\mu(y, w) \in \mathbf{Z}$  such that for any  $s \in S$  and any  $w \in W$  with  $sw > w$  we have  $c_s c_w = \sum_{y \in W; sy < y} \mu(y, w) c_y$ . By [KL2] we have  $\mu(y, w) \in \mathbf{N}$ .

**1.5.** Let  $\underline{\mathcal{H}} = \mathbf{C}(v) \otimes_{\mathcal{A}} \mathcal{H}$  where we use the obvious imbedding  $\mathcal{A} \rightarrow \mathbf{C}(v)$ ; we denote  $1 \otimes c_w$  again by  $c_w$ . Let  $\underline{\mathbf{J}} = \mathbf{C}(v) \otimes_{\mathbf{C}} \mathbf{J}$  where we use the obvious imbedding  $\mathbf{C} \rightarrow \mathbf{C}(v)$ . We have a homomorphism of  $\mathbf{C}(v)$ -algebras (with 1)  $\Psi : \underline{\mathcal{H}} \rightarrow \underline{\mathbf{J}}$  given by

$$\Psi(c_x) = \sum_{d \in \mathcal{D}, z \in W, d \sim z} h_{x, d, z} t_z$$

for all  $x \in W$ , see [L6, 18.9]. (Note that  $\Psi$  is in fact the composition of a homomorphism in *loc.cit.* with an automorphism of  $\underline{\mathcal{H}}$ .)

**1.6.** For any  $\Gamma \in L$  let  $S_{\Gamma}$  be the set of all  $t \in S$  such that  $rt < r$  for some (or equivalently any)  $r \in \Gamma$ .

We fix  $\Gamma \in L$ . Let  $\{\tilde{e}_{\Gamma'}; \Gamma' \in L\}$  be a  $\mathbf{C}$ -basis of  $M_{\Gamma}$  as in 1.3; we use the notation of 1.3. We shall view  $\mathbf{C}(v) \otimes M_{\Gamma}$  as an  $\underline{\mathcal{H}}$ -module via  $\Psi$ . Let  $s \in S$  and let  $\Gamma' \in L$ ; let  $\delta$  be the unique element in  $\Gamma' \cap \mathcal{D}$ . We show:

(a) If  $s \in S_{\Gamma'}$ , then  $\Psi(T_s) \tilde{e}_{\Gamma'} = v^2 \tilde{e}_{\Gamma'}$ .

(b) If  $s \notin S_{\Gamma'}$ , then

$$\Psi(T_s) \tilde{e}_{\Gamma'} = -\tilde{e}_{\Gamma'} + \sum_{\tilde{\Gamma} \in L; s \in S_{\tilde{\Gamma}}} f_{\tilde{\Gamma}, \Gamma'} v^{-1} \tilde{e}_{\tilde{\Gamma}}$$

where

$$f_{\tilde{\Gamma}, \Gamma'} = \sum_{u \in \Gamma' \cap \tilde{\Gamma}^{-1}} \mu(u, \delta) \lambda_{u, \Gamma', \tilde{\Gamma}} \in \mathbf{R}_{\geq 0}.$$

By definition, we have

$$\Psi(c_s)\tilde{e}_{\Gamma'} = \sum_{d \in \mathcal{D}, u \in c, d \sim u} h_{s,d,u} t_u \tilde{e}_{\Gamma'} = \sum_{\tilde{\Gamma} \in L} \sum_{d \in \mathcal{D}, u \in \Gamma' \cap \tilde{\Gamma}^{-1}, d \in \Gamma'} h_{s,d,u} \lambda_{u,\Gamma',\tilde{\Gamma}} \tilde{e}_{\tilde{\Gamma}}.$$

Since in the last sum we have  $d \in \Gamma'$  we see that we can assume that  $d = \delta$ . Thus we have

$$\Psi(c_s)\tilde{e}_{\Gamma'} = \sum_{\tilde{\Gamma} \in L} \sum_{u \in \Gamma' \cap \tilde{\Gamma}^{-1}} h_{s,\delta,u} \lambda_{u,\Gamma',\tilde{\Gamma}} \tilde{e}_{\tilde{\Gamma}}.$$

If  $s\delta < \delta$  (that is,  $s \in S_{\Gamma'}$ ) we have  $c_s c_\delta = (v + v^{-1})c_\delta$  hence  $h_{s,\delta,u}$  is  $(v + v^{-1})$  for  $u = \delta$  and is 0 for  $u \neq \delta$ ; hence in this case

$$\Psi(c_s)\tilde{e}_{\Gamma'} = (v + v^{-1})\tilde{e}_{\Gamma'};$$

(we use that  $\lambda_{\delta,\Gamma',\Gamma'} = 1$ .)

We now assume that  $s\delta > \delta$  (that is,  $s \notin S_{\Gamma'}$ ). In this case,  $h_{s,\delta,u}$  is  $\mu_{u,\delta}$  if  $su < u$  and is 0 if  $su > u$  (see 1.4); hence

$$\begin{aligned} \Psi(c_s)\tilde{e}_{\Gamma'} &= \sum_{\tilde{\Gamma} \in L} \sum_{u \in \Gamma' \cap \tilde{\Gamma}^{-1}; su < u} \mu(u, \delta) \lambda_{u,\Gamma',\tilde{\Gamma}} \tilde{e}_{\tilde{\Gamma}} \\ &= \sum_{\tilde{\Gamma} \in L; s \in S_{\tilde{\Gamma}}} \sum_{u \in \Gamma' \cap \tilde{\Gamma}^{-1}} \mu(u, \delta) \lambda_{u,\Gamma',\tilde{\Gamma}} \tilde{e}_{\tilde{\Gamma}}. \end{aligned}$$

Now (a),(b) follow.

Note that (a),(b) show that in the  $\underline{H}$ -module  $\mathbf{C}(v) \otimes M_\Gamma$  the generators  $T_s$  act with respect to the basis  $\{\tilde{e}_{\Gamma'}; \Gamma' \in L\}$  essentially by formulas which are those in a  $W$ -graph (in the sense of [KL1]) in which all labels are in  $\mathbf{R}_{\geq 0}$ .

**1.7.** In Section 4 we will give another proof of the existence part of 1.2(a) which also shows that  $\tilde{e}_{\Gamma'}$  in 1.3 can be chosen so that

- (i) each  $\tilde{e}_{\Gamma'}$  is a  $\mathbf{Z}_{>0}$ -linear combination of elements in  $\{t_x; x \in \Gamma \cap \Gamma'^{-1}\}$ ,
- (ii)  $\lambda_{u,\Gamma',\Gamma'_1} \in \mathbf{Z}_{\geq 0}$  (notation of 1.3).

In particular, with this choice of  $\tilde{e}_{\Gamma'}$ , the constants  $f_{\tilde{\Gamma},\Gamma'}$  in the " $W$ -graph formulas" in 1.6 are in  $\mathbf{Z}_{\geq 0}$ .

## 2. PROOF OF THEOREM 1.2

**2.1.** From [L6,§15] we see that, for  $x, y, u$  in  $W$  we have:

- (a)  $\gamma_{x,y,u} = \gamma_{y,u,x} = \gamma_{u,x,y},$
- (b)  $\gamma_{x,y,u} \neq 0 \implies x \sim y^{-1}, y \sim u^{-1}, u \sim x^{-1}.$

By [L6, 18.4(a)]:

- (c) for  $y, z$  in  $W$  we have  $y \sim z$  if and only if  $t_y t_{z^{-1}} \neq 0$ .

**2.2.** Let  $\Gamma, \Gamma', \tilde{\Gamma}, \tilde{\Gamma}'$  be in  $L$ . From 2.1(b) we deduce:

(a)  $\text{If } \Gamma \neq \tilde{\Gamma}', \text{ then } \mathbf{J}_{\Gamma \cap \Gamma'^{-1}} \mathbf{J}_{\tilde{\Gamma} \cap \tilde{\Gamma}'^{-1}} = 0,$

(b)  $\mathbf{J}_{\Gamma \cap \Gamma'^{-1}} \mathbf{J}_{\tilde{\Gamma} \cap \Gamma^{-1}} \subset \mathbf{J}_{\tilde{\Gamma} \cap \Gamma'^{-1}}.$

We show:

(c)  $\text{If } u \in \Gamma \cap \Gamma'^{-1}, \text{ then } t_u \mathbf{J}_{\tilde{\Gamma} \cap \Gamma^{-1}}^+ \subset \mathbf{J}_{\tilde{\Gamma} \cap \Gamma'^{-1}}^+.$

Let  $\xi = \sum_{y \in \tilde{\Gamma} \cap \Gamma^{-1}} f_y t_y \in \mathbf{J}_{\tilde{\Gamma} \cap \Gamma^{-1}}$  with  $f_y \in \mathbf{R}_{>0}$  for all  $y$ . We must show that  $t_u \xi \in \mathbf{J}_{\tilde{\Gamma} \cap \Gamma'^{-1}}^+$ ; it is enough to show that for any  $z \in \tilde{\Gamma} \cap \Gamma'^{-1}$  there exists  $y \in \tilde{\Gamma} \cap \Gamma^{-1}$  such that  $\gamma_{u,y,z^{-1}} \neq 0$  or that there exists  $y \in W$  such that  $\gamma_{z^{-1},u,y} \neq 0$  (see 2.1(a)); such  $y$  is automatically in  $\tilde{\Gamma} \cap \Gamma^{-1}$ . Hence it is enough to show that for any  $z \in \tilde{\Gamma} \cap \Gamma'^{-1}$  we have  $t_{z^{-1}} t_u \neq 0$ . This holds since  $z^{-1} \sim u^{-1}$  (see 2.1(c)).

From (c) we deduce

(d)  $\mathbf{J}_{\Gamma \cap \Gamma'^{-1}}^+ \mathbf{J}_{\tilde{\Gamma} \cap \Gamma^{-1}}^+ \subset \mathbf{J}_{\tilde{\Gamma} \cap \Gamma'^{-1}}^+.$

**2.3.** Let  $\Gamma \in L$ . For any  $\Gamma' \in L$  we define a  $\mathbf{C}$ -linear map  $T_{\Gamma'} : \mathbf{J}_{\Gamma \cap \Gamma'^{-1}} \rightarrow \mathbf{J}_{\Gamma \cap \Gamma'^{-1}}$  by

$$T_{\Gamma'}(t_x) = \sum_{y \in \Gamma \cap \Gamma^{-1}} t_x t_y = \sum_{y \in \Gamma \cap \Gamma^{-1}, z \in \Gamma} \gamma_{x,y,z^{-1}} t_z.$$

We show:

(a) *the matrix representing  $T_{\Gamma'}$  with respect to the basis  $\{t_w; w \in \Gamma \cap \Gamma'^{-1}\}$  has all entries in  $\mathbf{R}_{>0}$ .*

An equivalent statement is: for any  $x, z$  in  $\Gamma \cap \Gamma'^{-1}$ , the sum  $\sum_{y \in \Gamma \cap \Gamma^{-1}} \gamma_{x,y,z^{-1}}$  is  $> 0$ . Since  $\gamma_{x,y,z^{-1}} \in \mathbf{N}$  for all  $y$ , it is enough to show that for some  $y \in \Gamma \cap \Gamma^{-1}$  we have  $\gamma_{x,y,z^{-1}} \neq 0$  or equivalently (see 2.1(a)) that for some  $y \in W$  we have  $\gamma_{z^{-1},x,y} \neq 0$  (we then have automatically  $y \in \Gamma \cap \Gamma^{-1}$ ). Thus, it is enough to show that  $t_{z^{-1}} t_x \neq 0$ . This follows from 2.1(c) since  $z^{-1} \sim x^{-1}$ .

Applying Perron's theorem [Pe] to the matrix in (a) we see that there is a unique  $T_{\Gamma'}$ -stable positive line  $\mathcal{L}_{\Gamma, \Gamma'}$  in  $\mathbf{J}_{\Gamma \cap \Gamma'^{-1}}$  (the "Perron line").

Now let  $u \in \mathbf{c}$ ; we have,  $u \in \tilde{\Gamma} \cap \tilde{\Gamma}'^{-1}$  with  $\tilde{\Gamma}, \tilde{\Gamma}'$  in  $L$ . From 2.2(a), 2.2(d), we deduce

(b) If  $\tilde{\Gamma} \neq \Gamma'$ , then  $t_u \mathbf{J}_{\Gamma \cap \Gamma'^{-1}} = 0$  hence  $t_u \mathcal{L}_{\Gamma, \Gamma'} = 0$ ;

(c) if  $\tilde{\Gamma} = \Gamma'$ , then  $t_u \mathcal{L}_{\Gamma, \Gamma'}^+ \subset \mathbf{J}_{\Gamma \cap \tilde{\Gamma}'^{-1}}^+$ , hence  $t_u \mathcal{L}_{\Gamma, \Gamma'}$  is a positive line in  $\mathbf{J}_{\Gamma \cap \tilde{\Gamma}'^{-1}}$ .

In the setup of (c), we have  $t_u(T_{\Gamma'}(\xi)) = T_{\tilde{\Gamma}'}(t_u \xi)$  for any  $\xi \in \mathbf{J}_{\Gamma \cap \Gamma'^{-1}}$ . It follows that  $t_u \mathcal{L}_{\Gamma, \Gamma'}$  is a  $T_{\tilde{\Gamma}'}$ -stable line in  $\mathbf{J}_{\Gamma \cap \tilde{\Gamma}'^{-1}}$ . Thus,

(d) if  $\tilde{\Gamma} = \Gamma'$ , then  $t_u \mathcal{L}_{\Gamma, \Gamma'} = \mathcal{L}_{\Gamma, \tilde{\Gamma}'}$ .

We set  $\mathcal{M}_{\Gamma} = \bigoplus_{\Gamma' \in L} \mathcal{L}_{\Gamma, \Gamma'}$ . From (b), (d) we see that  $\mathcal{M}_{\Gamma}$  is a  $\mathbf{J}_{\mathbf{c}}$ -submodule of  $\mathbf{J}_{\Gamma}$ .

We now see that the existence part of 1.2(a) is proved: we can take  $M_{\Gamma} = \mathcal{M}_{\Gamma}$ .

**2.4.** Let  $\Gamma \in L$  and let  $M_\Gamma$  be any  $\mathbf{J}_c$ -submodule of  $\mathbf{J}_\Gamma$  for which property  $(\heartsuit)$  in 1.2(a) holds. We show that 1.2(b) holds for  $M_\Gamma$ . Let  $u \in \tilde{\Gamma} \cap \tilde{\Gamma}'^{-1}$  be as in 1.2(b) and let  $\Gamma' \in L$ . If  $\tilde{\Gamma} \neq \Gamma'$ , then  $t_u \mathbf{J}_{\Gamma \cap \Gamma'^{-1}} = 0$  hence  $t_u M_{\Gamma, \Gamma'} = 0$ . Now assume that  $\tilde{\Gamma} = \Gamma'$ . By 2.2(c), left multiplication by  $t_u$  maps  $\mathbf{J}_{\Gamma \cap \Gamma'^{-1}}^+$  into  $\mathbf{J}_{\Gamma \cap \tilde{\Gamma}'^{-1}}^+$  hence it maps any positive line in  $\mathbf{J}_{\Gamma \cap \Gamma'^{-1}}$  onto a positive line in  $\mathbf{J}_{\Gamma \cap \tilde{\Gamma}'^{-1}}$ . In particular, it maps  $M_{\Gamma, \Gamma'}$  onto a line in  $\mathbf{J}_{\Gamma \cap \tilde{\Gamma}'^{-1}}$ , which, being also contained in  $M_\Gamma$ , must be equal to  $M_{\Gamma, \tilde{\Gamma}'}$ ; moreover, it maps  $M_{\Gamma, \Gamma'}^+$  into  $\mathbf{J}_{\Gamma \cap \tilde{\Gamma}'^{-1}}^+$  hence onto  $M_{\Gamma, \tilde{\Gamma}'}^+$ . Thus, 1.2(b) holds for  $M_\Gamma$ .

We now choose a basis  $\{\tilde{e}_{\Gamma'}; \Gamma' \in L\}$  of  $M_\Gamma$  such that  $\tilde{e}_{\Gamma'} \in M_{\Gamma, \Gamma'}^+$  for any  $\Gamma' \in L$ ; then for any  $u \in c$ , the matrix of the  $t_u$ -action on  $M_\Gamma$  in this basis has entries in  $\mathbf{R}_{\geq 0}$ . Thus,

(a)  $\text{tr}(t_u, M_\Gamma) \in \mathbf{R}_{\geq 0}$  for all  $u \in c$ .

We show:

(b) *the  $\mathbf{C}$ -linear map  $\nu : \mathbf{J}_c \rightarrow \text{End}_{\mathbf{C}}(M_\Gamma)$  given by the  $\mathbf{J}_c$ -module structure on  $M_\Gamma$  is surjective.*

It is enough to show that for any  $\Gamma', \tilde{\Gamma}'$  in  $L$  there exists  $u \in c$  such that  $\nu(t_u)$  carries the line  $M_{\Gamma, \Gamma'}$  onto the line  $M_{\Gamma, \tilde{\Gamma}'}$  and carries the line  $M_{\Gamma, \Gamma''}$  (where  $\Gamma'' \in L$ ,  $\Gamma'' \neq \Gamma'$ ) to zero. Note that any  $u \in \Gamma' \cap \tilde{\Gamma}'^{-1}$  has the required properties. This proves (b).

It follows that the  $\mathbf{J}_c$ -module  $M_\Gamma$  is simple. We show:

(c) *Assume that  $M'$  is any simple  $\mathbf{J}_c$ -module such that  $\text{tr}(t_u, M') \in \mathbf{R}_{\geq 0}$  for all  $u \in c$ . Then  $M'$  is isomorphic to  $M_\Gamma$ .*

Assume that this is not so. We use the orthogonality formula

$$\sum_{u \in c} \text{tr}(t_u, M_\Gamma) \text{tr}(t_{u^{-1}}, M') = 0,$$

which is a special case of [L6, 19.2(e)] (taking into account [L6, 20.1(b)] and using that  $u \mapsto u^{-1}$  maps  $c$  into  $c$ ). Since each term in the last sum is in  $\mathbf{R}_{\geq 0}$ , it follows that each term in the last sum is 0. In particular, we have  $\text{tr}(t_d, M_\Gamma) \text{tr}(t_d, M') = 0$  for any  $d \in \mathcal{D}_c$ . We show that for any  $d \in \mathcal{D}_c$  we have  $\text{tr}(t_d, M_\Gamma) \in \mathbf{R}_{> 0}$ . Using the basis of  $M_\Gamma$  employed in the proof of (a), it is enough to show that some diagonal entry of the matrix of the  $t_d$ -action in this basis is  $\neq 0$  (all entries are in  $\mathbf{R}_{\geq 0}$ ). We have  $d \in \Gamma'$  for a unique  $\Gamma' \in L$ ; then  $t_d M_{\Gamma, \Gamma'}^+ = M_{\Gamma, \Gamma'}^+$  and the desired property holds.

From  $\text{tr}(t_d, M_\Gamma) \text{tr}(t_d, M') = 0$  and  $\text{tr}(t_d, M_\Gamma) \in \mathbf{R}_{> 0}$  we deduce that  $\text{tr}(t_d, M') = 0$  for any  $d \in \mathcal{D}_c$ . Since  $\sum_{d \in \mathcal{D}_c} t_d$  is the unit element  $1_c$  of  $\mathbf{J}_c$ , it follows that  $\text{tr}(1_c, M') = 0$ . This is a contradiction. This proves (c). ■

Let  $I$  be the simple ideal of  $\mathbf{J}_c$  such that  $I M_\Gamma \neq 0$ . It is a  $\mathbf{C}$ -vector space of dimension  $N^2$  where  $N$  is the number of elements in  $L$ . If  $\tilde{\Gamma} \in L$ , then  $\mathcal{M}_{\tilde{\Gamma}}$  is a simple  $\mathbf{J}_c$ -module such that  $\text{tr}(t_u, \mathcal{M}_{\tilde{\Gamma}}) \in \mathbf{R}_{\geq 0}$  for all  $u \in c$  (we use (a) with  $M_\Gamma$  replaced by  $\mathcal{M}_{\tilde{\Gamma}}$ ); hence, by (c), we have  $\mathcal{M}_{\tilde{\Gamma}} \cong M_\Gamma$  as  $\mathbf{J}_c$ -modules. In particular,

the isomorphism class of  $\mathcal{M}_{\tilde{\Gamma}}$  is independent of  $\tilde{\Gamma}$ . We see that the (necessarily direct) sum  $\sum_{\tilde{\Gamma} \in L} \mathcal{M}_{\tilde{\Gamma}}$  is contained in  $I$  and has dimension  $N^2$  hence it is equal to  $I$ ; we also see that  $M_{\Gamma} \subset I$  and, taking intersections with  $\mathbf{J}_{\Gamma}$ , we see that  $M_{\Gamma} \subset \mathcal{M}_{\Gamma}$ , hence  $M_{\Gamma} = \mathcal{M}_{\Gamma}$  (since  $\dim M_{\Gamma} = \dim \mathcal{M}_{\Gamma} = N$ ). We now see that the uniqueness part of 1.2(a) is proved. Note that 1.2(b), 1.2(c), 1.2(d) are also proved and we have  $\mathbf{I} = I$ .

**2.5.** We prove 1.2(e). In the setup of (e), if  $\Gamma \neq \tilde{\Gamma}'$  then, using 2.2(a), we have  $M_{\Gamma, \Gamma'} M_{\tilde{\Gamma}, \tilde{\Gamma}'} = 0$ . Assume now that  $\Gamma = \tilde{\Gamma}'$ . Using 2.2(d), we see that  $M_{\Gamma, \Gamma'}^+ M_{\tilde{\Gamma}, \tilde{\Gamma}'}^+$  is contained in  $\mathbf{J}_{\tilde{\Gamma} \cap \Gamma'^{-1}}^+$ ; it is also contained in  $\mathbf{I}$  (since  $\mathbf{I}$  is closed under multiplication), hence it is contained in  $\mathbf{I} \cap \mathbf{J}_{\tilde{\Gamma} \cap \Gamma'^{-1}}^+ = M_{\tilde{\Gamma}, \Gamma'}^+$ . Thus, multiplication restricts to a map  $M_{\Gamma, \Gamma'}^+ \times M_{\tilde{\Gamma}, \tilde{\Gamma}'}^+ \rightarrow M_{\tilde{\Gamma}, \Gamma'}^+$ . This map is necessarily surjective since  $M_{\tilde{\Gamma}, \Gamma'}^+$  is a single orbit of  $\mathbf{R}_{>0}$  under scalar multiplication. This implies that the linear map between lines  $M_{\Gamma, \Gamma'} \otimes M_{\tilde{\Gamma}, \tilde{\Gamma}'} \rightarrow M_{\tilde{\Gamma}, \Gamma'}$  is an isomorphism. This proves 1.2(e).

**2.6.** We prove 1.2(f). Any element  $\xi \in \mathbf{J}_{\mathbf{c}}$  defines a linear map  ${}^t(\theta(\xi)) : M_{\Gamma}^* \rightarrow M_{\Gamma}^*$  where  $M_{\Gamma}^*$  denotes the dual space and  ${}^t$  denotes the transpose. This defines a  $\mathbf{J}_{\mathbf{c}}$ -module structure on  $M_{\Gamma}^*$  such that for any  $x \in \mathbf{c}$  we have  $\text{tr}(t_x, M_{\Gamma}^*) = \text{tr}(t_{x^{-1}}, M_{\Gamma})$ . By the argument in [L6, 20.13(a)],  $\text{tr}(t_{x^{-1}}, M_{\Gamma})$  is the complex conjugate of  $\text{tr}(t_x, M_{\Gamma})$ . But the last trace is a real number, so that  $\text{tr}(t_x, M_{\Gamma}^*) = \text{tr}(t_x, M_{\Gamma})$ . It follows that  $M_{\Gamma}^* \cong M_{\Gamma}$  as  $\mathbf{J}_{\mathbf{c}}$ -modules. From the definitions, the simple two-sided ideal  $\mathbf{I}'$  of  $\mathbf{J}_{\mathbf{c}}$  such that  $\mathbf{I}' M_{\Gamma}^* \neq 0$  satisfies  $\mathbf{I}' = \theta(\mathbf{I})$ . It follows that  $\theta(\mathbf{I}) = \mathbf{I}$ . Since  $\mathbf{I} = \oplus_{\tilde{\Gamma}, \tilde{\Gamma}' \text{ in } L} M_{\tilde{\Gamma}, \tilde{\Gamma}'}$  and  $\theta(\mathbf{J}_{\Gamma, \Gamma'}) = \mathbf{J}_{\Gamma', \Gamma}$ , it follows that

$$\theta(M_{\Gamma, \Gamma'}) \subset \mathbf{J}_{\Gamma', \Gamma} \cap \oplus_{\tilde{\Gamma}, \tilde{\Gamma}' \text{ in } L} M_{\tilde{\Gamma}, \tilde{\Gamma}'} = M_{\Gamma', \Gamma}.$$

Since  $\theta$  is a vector space isomorphism, it follows that  $\theta(M_{\Gamma, \Gamma'}) = M_{\Gamma', \Gamma}$ . Note that  $\theta(\mathbf{J}_{\Gamma, \Gamma'}^+) = \mathbf{J}_{\Gamma', \Gamma}^+$ ; hence

$$\theta(M_{\Gamma, \Gamma'}^+) \subset \mathbf{J}_{\Gamma', \Gamma}^+ \cap M_{\Gamma', \Gamma} = M_{\Gamma', \Gamma}^+.$$

This forces the equality  $\theta(M_{\Gamma, \Gamma'}^+) = M_{\Gamma', \Gamma}^+$  (since  $M_{\Gamma', \Gamma}^+$  is a single orbit of  $\mathbf{R}_{>0}$  under scalar multiplication). This proves 1.2(f). Theorem 1.2 is proved.

**2.7.** After an earlier version of this paper was posted, P. Etingof told me that the line  $M_{\Gamma, \Gamma'}$  in 1.2 is the same as the line associated in [EGNO, 3.4.4] to the right  $\mathbf{J}_{\Gamma \cap \Gamma^{-1}}$ -module  $\mathbf{J}_{\Gamma \cap \Gamma'^{-1}}$  (viewed as a based module over a based ring) that is, the unique positive line  $\mathfrak{L}$  in  $\mathbf{J}_{\Gamma \cap \Gamma'^{-1}}$  such that  $\mathfrak{L}$  is a right  $\mathbf{J}_{\Gamma \cap \Gamma^{-1}}$ -submodule. (The discussion in *loc.cit.* concerns left (instead of right) indecomposable based modules over a fusion ring.) Indeed, from the definitions we see that  $\mathfrak{L}$  must be the same as  $\mathcal{L}_{\Gamma, \Gamma'}$  in 2.3, hence the same as  $M_{\Gamma, \Gamma'}$ .

### 3. SPECIAL REPRESENTATIONS

**3.1.** When  $\mathcal{H}$  is tensored with  $\mathbf{C}$  (using the ring homomorphism  $\mathcal{A} \rightarrow \mathbf{C}$ ,  $v \mapsto 1$ ), then it becomes  $\mathbf{C}[W]$ , the group algebra of  $W$ . (For  $w \in W$  we have  $1 \otimes T_w = w \in$

$\mathbf{C}[W]$ ; we denote  $1 \otimes c_w$  again by  $c_w$ .) We have a homomorphism of  $\mathbf{C}$ -algebras (with 1)  $\psi : \mathbf{C}[W] \rightarrow \mathbf{J}$  given by

$$\psi(c_x) = \sum_{d \in \mathcal{D}, z \in W, d \sim z} h_{x,d,z}|_{v=1} t_z$$

for all  $x \in W$ , see [L6, 18.9]; this is an isomorphism, see [L6, 20.1]. For example, if  $W = \{1, s\}$  is of type  $A_1$  we have  $\psi(c_1) = t_1 + t_s$ ,  $\psi(c_s) = 2t_s$ ; hence  $\psi(1) = t_1 + t_s$ ,  $\psi(s) = -t_1 + t_s$ .

For each  $E \in \text{Irr}W$  let  $E_\infty$  be the simple  $\mathbf{J}$ -module corresponding to  $E$  under  $\psi$  and let  $\mathbf{c}_E$  be the unique two-sided cell of  $W$  such that  $\mathbf{J}_{\mathbf{c}_E} E \neq 0$ . (Note that  $E = E_\infty$  as  $\mathbf{C}$ -vector spaces.) Let  $\text{Irr}^{\mathbf{c}}W = \{E \in \text{Irr}W; \mathbf{c}_E = \mathbf{c}\}$  and let  $a' = a(xw_0)$  for any  $x \in \mathbf{c}$ , where  $w_0$  is the longest element of  $W$ .

For any  $k \in \mathbf{N}$  let  $\mathfrak{S}^k$  be the  $k$ -th symmetric power of the reflection representation of  $W$ , viewed as a representation of  $W$  in an obvious way. For  $E \in \text{Irr}W$  let  $b_E$  be the smallest integer  $k \geq 0$  such that  $E$  is a constituent of  $\mathfrak{S}^k$ . Now for any  $E \in \text{Irr}^{\mathbf{c}}W$  we have  $b_E \geq a'$  and there is a unique  $E \in \text{Irr}^{\mathbf{c}}W$  such that  $b_E = a'$ ; this  $E$  is denoted by  $E^{\mathbf{c}}$  and is called the *special representation* associated to  $\mathbf{c}$ . (This is a reformulation of the definition of special representations given in [L1].)

**Theorem 3.2.** *In the setup of Theorem 1.2, for any  $\Gamma \in L$ , we have  $M_\Gamma \cong E_\infty^{\mathbf{c}}$  as  $\mathbf{J}_{\mathbf{c}}$ -modules.*

We give two proofs; one is contained in 3.3, 3.4, 3.5. The other is given in 3.4, 3.6.

**3.3.** In this subsection we assume that  $W$  is of type  $A, B$  or  $D$ .

Let  $\Gamma \in L$ . For any  $\Gamma' \in L$  we set

$$\epsilon_{\Gamma'} = \sum_{z \in \Gamma \cap \Gamma'^{-1}} t_z \in \mathbf{J}_{\Gamma \cap \Gamma'^{-1}}^+.$$

By [L9, 4.8(b)],  $\{\epsilon_{\Gamma'}; \Gamma' \in L\}$  is a  $\mathbf{C}$ -basis of the unique  $\mathbf{J}$ -submodule of  $\mathbf{J}_\Gamma$  isomorphic to  $E_\infty^{\mathbf{c}}$ . By the uniqueness part of 1.2(a) this  $\mathbf{J}$ -submodule of  $\mathbf{J}_\Gamma$  (viewed as a  $\mathbf{J}_{\mathbf{c}}$ -module) must be the same as  $M_\Gamma$  in 1.2(a). We see that in this case,  $M_\Gamma$  is isomorphic to  $E_\infty^{\mathbf{c}}$  and  $M_{\Gamma, \Gamma'}$  is the line spanned by  $\epsilon_{\Gamma'}$ . In particular, 3.2 holds in our case.

**3.4.** In this subsection we assume that  $\mathbf{c}$  is such that  $\text{Irr}^{\mathbf{c}}W$  consists of exactly 2 irreducible representations. In this case,  $W$  is of type  $E_7$  (resp.  $E_8$ ) and the 2 irreducible representations in  $\text{Irr}^{\mathbf{c}}W$  have degree 512 (resp. 4096). Let  $\Gamma \in L$  and let  $d \in \mathcal{D} \cap \Gamma$ . The  $\mathbf{C}$ -linear map  $r : \mathbf{J}_\Gamma \rightarrow \mathbf{J}_\Gamma$  given by left multiplication by  $(-1)^{l(d)}\psi(w_0)$  is in fact  $\mathbf{J}$ -linear (since  $w_0$  is central in  $W$ ) and  $r(t_x) = t_{x^*}$  for any  $x \in \Gamma$ , where  $x \mapsto x^*$  is a certain fixed point free involution of  $\Gamma$ , see [L7]. Then  $\mathbf{J}_\Gamma^1 = \{\xi \in \mathbf{J}_\Gamma; r(\xi) = \xi\}$  is a simple  $\mathbf{J}_{\mathbf{c}}$ -submodule of  $\mathbf{J}_\Gamma$  with  $\mathbf{C}$ -basis  $\{t_x + t_{x^*}; x \in \Gamma_1\}$  where  $\Gamma_1$  is a set of representatives for the orbits of  $x \mapsto x^*$  on  $\Gamma$ . Note that, if  $x \in \Gamma$ , then  $\{x, x^*\}$  is the intersection of  $\Gamma$  with the inverse



of a left cell  $\Gamma' \in L$ ; hence  $t_x + t_{x^*} \in \mathbf{J}_{\Gamma \cap \Gamma'^{-1}}^+$ . By the uniqueness part of 1.2(a), we must have  $M_\Gamma = \mathbf{J}_\Gamma^1$  and that for any  $\Gamma' \in L$ ,  $M_{\Gamma, \Gamma'}$  is the line spanned by  $\sum_{x \in \Gamma \cap \Gamma'^{-1}} t_x$ .

Now let  $E \in \text{Irr}^c W$  be such that  $E_\infty = \mathbf{J}_\Gamma^1$ . From the definitions we have  $\text{tr}((-1)^{l(d)} w_0, E) = |\Gamma_1| = \dim E$ . Hence, if  $\epsilon = \pm 1$  is the scalar by which  $w_0$  acts on  $E$ , then  $(-1)^{l(d)} \epsilon = 1$ . We have  $l(d) = a(d) \pmod{2}$ ; hence  $\epsilon = (-1)^{a(d)}$ . But this equality characterizes the special representation in  $\text{Irr}^c W$  (the special representation satisfies it, the nonspecial representation doesn't satisfy it). We see that  $M_\Gamma = \mathbf{J}_\Gamma^1 \cong E_\infty$ . In particular, 3.2 holds in our case.

**3.5.** In this subsection we assume that  $W$  is of exceptional type, but that  $\mathbf{c}$  is not as in 3.4. In this case,  $E^c$  is the only representation in  $\text{Irr}^c W$  of dimension equal to  $|L|$ ; since  $M_\Gamma$  (in 1.2(a)) has dimension equal to  $|L|$ , it follows that  $M_\Gamma = \mathbf{J}_\Gamma^1 \cong E_\infty^c$ . In particular, 3.2 holds in our case. This completes the proof of Theorem 3.2.

**3.6.** In this subsection we give a second proof of Theorem 3.2 assuming that  $\mathbf{c}$  is not as in 3.4. Let  $a = a(w)$  for any  $w \in \mathbf{c}$ . Let  $\Gamma \in L$ . Let  $X = \sum_{w \in W} v^{-l(w)} T_w \in \underline{\mathcal{H}}$ . We can view  $\mathbf{C}(v) \otimes \mathbf{J}_\Gamma$  and  $\mathbf{C}(v) \otimes M_\Gamma$  as  $\underline{\mathcal{H}}$ -modules via  $\Psi$  in 1.5. By [L9, 4.6], for any  $x \in \Gamma$  we have

$$X t_x = v^a \sum_{z \in \mathbf{c}} t_z t_x \pmod{\sum_{i < a} v^i \mathbf{J}_\Gamma}.$$

By an argument as in the proof of 2.2(c), we see that  $\sum_{z \in \mathbf{c}} t_z t_x \in \mathbf{J}_\Gamma^+$ . It follows that if  $\Gamma' \in L$  and  $\xi \in M_{\Gamma, \Gamma'}$  then

$$X \xi = v^a \xi' \pmod{\sum_{i < a} v^i \mathbf{J}_\Gamma}$$

where  $\xi' \in \mathbf{J}_\Gamma^+$ . In particular, we have  $X \xi \neq 0$ . Thus,  $X(\mathbf{C}(v) \otimes M_\Gamma) \neq 0$ . Using this and Theorem 4.2 in [L9] we deduce that the simple  $\underline{\mathcal{H}}$ -module  $\mathbf{C}(v) \otimes M_\Gamma$  is a constituent of the "involution module"  $M$  in [L9, 0.1] (with  $\mathbf{Q}(u)$  replaced by  $\mathbf{C}(v)$ ). According to [L8] if a simple  $\underline{\mathcal{H}}$ -module appears in  $\mathbf{C}(v) \otimes \mathbf{J}_{\Gamma'}$  for every  $\Gamma' \in L$  and it appears in  $M$ , then that  $\underline{\mathcal{H}}$ -module corresponds to  $E^c$ . We deduce that  $M_\Gamma \cong E_\infty^c$ . This completes the second proof of Theorem 3.2, assuming that  $\mathbf{c}$  is not as in 3.4.

#### 4. EQUIVARIANT VECTOR BUNDLES

**4.1.** In this section we fix a reductive, not necessarily connected algebraic group  $G$  over  $\mathbf{C}$  acting on a finite set  $X$ . Let  $G \backslash X$  be the set of  $G$ -orbits on  $X$ . Representations of reductive groups over  $\mathbf{C}$  are always assumed to be of finite dimension over  $\mathbf{C}$  and algebraic. For  $x \in X$  let  $G_x = \{g \in G; gx = x\}$ . Now  $G$  acts diagonally on  $X \times X$  and we can consider the Grothendieck group  $K_G(X \times X)$  of  $G$ -equivariant complex vector bundles ( $G$ -eq.v.b.) on  $X \times X$ . This is an (associative) ring with

1 under convolution, denoted by  $*$  (see [L3, 2.2], [L4, 10.2]). For a  $G$ -eq.v.b.  $V$  on  $X \times X$  we denote by  $V_{x,y}$  the fibre of  $V$  at  $(x, y) \in X \times X$ . Let  $B$  be the set of pairs  $(\Omega, \rho)$  where  $\Omega$  is a  $G$ -orbit  $\Omega$  in  $X \times X$  and  $\rho$  is an irreducible representation of  $G_\Omega$  (the isotropy group of a point  $(x, y) \in \Omega$ ). For any  $(\Omega, \rho) \in B$  we denote by  $V^{\Omega, \rho}$  the  $G$ -eq.v.b. on  $X \times X$  such that  $V_{\Omega, \rho}|_{X \times X - \Omega} = 0$  and the action of  $G_\Omega$  on  $V_{x,y}^{\Omega, \rho}$  is equivalent to  $\rho$ . Now  $\{V^{\Omega, \rho}; (\Omega, \rho) \in B\}$  is a  $\mathbf{Z}$ -basis of  $K_G(X \times X)$ . Let  $\mathbf{K}_G(X \times X) = \mathbf{C} \otimes K_G(X \times X)$ , viewed as a  $\mathbf{C}$ -algebra.

**4.2.** In this subsection we assume that  $G$  is finite. Let  $\omega, \omega'$  be in  $G \backslash X$ . Let  $V^{\omega, \omega'}$  be the  $G$ -eq.v.b. on  $X \times X$  such that  $V_{a,b}^{\omega, \omega'} = \mathbf{C}[G]$  if  $(a, b) \in \omega \times \omega'$ ,  $V_{a,b}^{\omega, \omega'} = 0$  if  $(a, b) \notin \omega \times \omega'$ . (Here  $\mathbf{C}[G]$  is the left regular representation of  $G$ .) The  $G$ -action  $g : V_{a,b}^{\omega, \omega'} \rightarrow V_{ga, gb}^{\omega, \omega'}$  is left translation by  $g$  on  $\mathbf{C}[G]$  (if  $(a, b) \in \omega \times \omega'$ ) and is 0 if  $(a, b) \notin \omega \times \omega'$ . We show:

(a) *Let  $(\Omega, \rho) \in B$ ; we have  $\Omega \subset \omega_1 \times \omega'_1$  where  $\omega_1, \omega'_1$  are in  $G \backslash X$ . Then  $U' := V^{\Omega, \rho} * V^{\omega, \omega'}$  is isomorphic to a direct sum of copies of the single  $G$ -eq.v.b.  $V^{\omega_1, \omega'_1}$ . More precisely, if  $\omega'_1 \neq \omega$ , we have  $U' = 0$ ; if  $\omega'_1 = \omega$ , we have  $U' = V_{\omega_1, \omega'}^{\oplus (\dim \rho |\Omega| |\omega_1|^{-1})}$ .*

For  $(a, b) \in X \times X$  we have

$$U'_{a,b} = \bigoplus_{z \in \omega; (a,z) \in \Omega} V_{a,z}^{\Omega, \rho} \otimes \mathbf{C}[G] \text{ if } b \in \omega',$$

$$U'_{a,b} = 0 \text{ if } b \notin \omega'.$$

Thus the support of  $U'$  is contained in  $\omega_1 \times \omega'$  and  $U' = 0$  unless  $\omega'_1 = \omega$ . We now assume that  $\omega'_1 = \omega$  and  $(a, b) \in \omega_1 \times \omega'$ . Then  $U'_{a,b} = \bigoplus_{z; (a,z) \in \Omega} V_{a,z}^{\Omega, \rho} \otimes \mathbf{C}[G]$ . We have  $\dim U'_{a,b} = d|G||\Omega|/|\omega_1|$ . We show:

(b) *as a  $G_a \cap G_b$ -module,  $U'_{a,b}$  is a multiple of the regular representation.*  
Let  $\sigma_1, \dots, \sigma_k$  be the various  $G_a \cap G_b$ -orbits contained in  $\omega'$ . We have  $U'_{a,b} = \bigoplus_{i=1}^k R_i$ , where  $R_i = \bigoplus_{z \in \sigma_i} V_{a,z}^{\Omega, \rho} \otimes \mathbf{C}[G]$ . We pick  $z_i \in \sigma_i$ . Now

$$R_i = \text{ind}_{G_a \cap G_b \cap G_{z_i}}^{G_a \cap G_b} (A \otimes B)$$

where

$$A = \text{res}_{G_a \cap G_b \cap G_{z_i}}^{G_a \cap G_{z_i}} (V_{a,z_i}^{\Omega, \rho}), \quad B = \text{res}_{G_a \cap G_b \cap G_{z_i}}^{G_{z_i} \cap G_b} (\mathbf{C}[G]).$$

It is enough to show that  $R_i$  is a multiple of the regular representation of  $G_a \cap G_b$ . Since  $R_i$  is induced, it is enough to show that  $A \otimes B$  is a multiple of the regular representation of  $G_a \cap G_b \cap G_{z_i}$ . It is also enough to show that  $B$  is a multiple of the regular representation of  $G_a \cap G_b \cap G_{z_i}$ . This follows from the fact that  $\mathbf{C}[G]$  is a multiple of the regular representation of  $G_{z_i} \cap G_b$ . This proves (b). Now (a) follows.

Note that in  $\mathbf{K}_G(X \times X)$  we have

$$(c) \quad V^{\omega, \omega'} = \sum_{(\Omega, \rho) \in B; \Omega \subset \omega \times \omega'} \dim \rho |\Omega| V^{\Omega, \rho}.$$

**4.3.** We now drop the assumption (in 4.2) that  $G$  is finite. Let  $\hat{\mathbf{K}}_G(X \times X)$  be the  $\mathbf{C}$ -vector space consisting of formal (possibly infinite) linear combinations  $\sum_{(\Omega, \rho) \in B} f_{\Omega, \rho} V^{\Omega, \rho}$  where  $f_{\Omega, \rho} \in \mathbf{C}$ . The left  $\mathbf{K}_G(X \times X)$ -module structure on  $\mathbf{K}_G(X \times X)$  given by left multiplication extends naturally to a left  $\mathbf{K}_G(X \times X)$ -module structure on  $\hat{\mathbf{K}}_G(X \times X)$ . If  $\omega, \omega'$  are as in  $G \setminus X$  then we can define  $V^{\omega, \omega'} \in \hat{\mathbf{K}}_G(X \times X)$  by the sum 4.2(c) (which is now a possibly infinite sum); we set  $\bar{V}^{\omega, \omega'} = |\omega|^{-1} V^{\omega, \omega'}$  so that

$$(a) \quad \bar{V}^{\omega, \omega'} = \sum_{(\Omega, \rho) \in B; \Omega \subset \omega \times \omega'} \dim \rho |\Omega| |\omega|^{-1} V^{\Omega, \rho}.$$

Now formula 4.2(a) extends to the present case as follows. Let  $(Om, \rho) \in B$ ; we have  $\Omega \subset \omega_1 \times \omega'_1$  where  $\omega_1, \omega'_1$  are  $G$ -orbits in  $X$ . Then

$$(b) \quad V^{\Omega, \rho} V^{\omega, \omega'} = N V^{\omega_1, \omega'_1}$$

where  $N \in \mathbf{Z}$  is 0 if  $\omega'_1 \neq \omega$  and  $N = \dim \rho |\Omega| |\omega_1|^{-1}$  if  $\omega'_1 = \omega$ . Hence

$$(c) \quad V^{\Omega, \rho} \bar{V}^{\omega, \omega'} = N' \bar{V}^{\omega_1, \omega'_1}$$

where  $N' \in \mathbf{Z}$  is 0 if  $\omega'_1 \neq \omega$  and  $N' = \dim \rho |\Omega| |\omega|^{-1}$  if  $\omega'_1 = \omega$ .

Let  $\omega' \in G \setminus X$ . Let  $\tilde{R}_{\omega'}$  be the subspace of  $\hat{\mathbf{K}}_H(X \times X)$  consisting of formal (possibly infinite) linear combinations  $\sum_{(\Omega, \rho) \in B; pr_2 \Omega = \omega'} f_{\Omega, \rho} V^{\Omega, \rho}$  with  $f_{\Omega, \rho} \in \mathbf{C}$ . Here  $pr_2 : X \times X \rightarrow X$  is the second projection. Note that  $\tilde{R}_{\omega'}$  is a  $\mathbf{K}_G(X \times X)$ -submodule of  $\hat{\mathbf{K}}_H(X \times X)$ .

Let  $R_{\omega'}$  be the subspace of  $\hat{\mathbf{K}}_G(X \times X)$  with basis formed by the elements  $\bar{V}^{\omega, \omega'}$  for various  $\omega \in G \setminus X$ . Using (c) we see that  $R_{\omega'}$  is a (simple)  $\mathbf{K}_G(X \times X)$ -submodule of  $\hat{\mathbf{K}}_G(X \times X)$ ; we have  $R_{\omega'} \subset \tilde{R}_{\omega'}$ . Using (c) we see also that if  $\omega'' \in G \setminus X$  then  $\bar{V}^{\omega, \omega'} \mapsto \bar{V}^{\omega, \omega''}$  defines an isomorphism of  $\mathbf{K}_G(X \times X)$ -modules  $R_{\omega'} \xrightarrow{\sim} R_{\omega''}$ . Hence the isomorphism class of the  $\mathbf{K}_G(X \times X)$ -module  $R_{\omega'}$  is independent of the choice of  $\omega'$ .

**4.4.** We now assume that  $G$  is the finite group associated to  $\mathbf{c}$  in [L2] and that  $X$  is the finite  $G$ -set  $\oplus_{\Gamma \in L} G/H_{\Gamma}$  where  $H_{\Gamma}$  is the subgroup of  $G$  defined in [L3, 3.8]. In this case we have  $\hat{\mathbf{K}}_G(X \times X) = \mathbf{K}_G(X \times X)$ . By a conjecture in [L3, 3.15], proved in [BFO], there exists an isomorphism of  $\mathbf{C}$ -algebras  $\chi : \mathbf{K}_G(X \times X) \xrightarrow{\sim} \mathbf{J}_{\mathbf{c}}$  carrying the basis  $(V^{\Omega, \rho})$  of  $\mathbf{K}_G(X \times X)$  onto the basis  $\{t_x; x \in \mathbf{c}\}$  of  $\mathbf{J}_{\mathbf{c}}$ . Under  $\chi$ , the left ideal  $\mathbf{J}_{\Gamma}$  of  $\mathbf{J}_{\mathbf{c}}$  (for  $\Gamma \in L$ ) corresponds to the left ideal  $\tilde{R}_{\omega'}$  of  $\mathbf{K}_G(X \times X)$  where  $\omega' \in G \setminus X$  corresponds to  $\Gamma$ , and the basis  $\{t_x; x \in \Gamma\}$  of  $\mathbf{J}_{\Gamma}$  corresponds to the intersection of the basis  $(V^{\Omega, \rho})$  of  $\mathbf{K}_G(X \times X)$  with  $\tilde{R}_{\omega'}$ . The basis of  $R_{\omega'}$  formed by the elements  $\bar{V}^{\omega, \omega'}$  corresponds to a family of elements  $\{e_{\Gamma'}; \Gamma' \in L\}$  in  $\mathbf{J}_{\Gamma}$ .

From 4.3(a) we see that  $e_{\Gamma'} \in \mathbf{J}_{\Gamma \cap \Gamma'^{-1}}^+$  for any  $\Gamma' \in L$  (in fact the coefficients of the various  $t_x, x \in \Gamma \cap \Gamma'^{-1}$  are in  $\mathbf{Z}_{>0}$ ) and from 4.3(c) we see that for  $u \in \mathbf{c}$  the product  $t_u e_{\Gamma'}$  is a  $\mathbf{Z}_{\geq 0}$  multiple of an element  $e_{\Gamma''}$ . We see that the  $\mathbf{C}$ -subspace of  $\mathbf{J}_{\Gamma}$  spanned by  $\{e_{\Gamma'}; \Gamma' \in L\}$  satisfies property  $(\heartsuit)$  in 1.2(a) hence it is equal to  $M_{\Gamma}$ . This provides another proof in our case for the existence part of 1.2(a), with the additional integrality properties in 1.7.

## 5. FINAL REMARKS

**5.1.** Theorem 1.2 and its proof remain valid if  $W$  is replaced by an affine Weyl group (with  $\mathbf{c}$  assumed to be finite) or by a finite Coxeter group; in the last case we use the positivity property of  $h_{x,y,z}$  established in [EW]. In these cases, the simple  $\mathbf{J}_{\mathbf{c}}$ -module given by Theorem 1.2 will be called the special  $\mathbf{J}_{\mathbf{c}}$ -module.

**5.2.** Assume now that  $W$  is an (irreducible) affine Weyl group and that  $\mathbf{c}$  is a not necessarily finite two-sided cell of  $W$ . We denote again by  $L$  the set of left cells of  $W$  that are contained in  $\mathbf{c}$ ; this is a finite set. Then the  $\mathbf{C}$ -algebra  $\mathbf{J}_{\mathbf{c}}$  with its basis  $\{t_x; x \in \mathbf{c}\}$  is defined. Let  $\hat{\mathbf{J}}_{\mathbf{c}}$  be the set of formal (possibly infinite) linear combinations  $\sum_{u \in \mathbf{c}} f_u t_u$  where  $f_u \in \mathbf{C}$ . This is naturally a left  $\mathbf{J}_{\mathbf{c}}$ -module. For any subset  $X$  of  $\mathbf{c}$  let  $\hat{\mathbf{J}}_X$  be the set of all  $\sum_{u \in \mathbf{c}} f_u t_u \in \hat{\mathbf{J}}_{\mathbf{c}}$  such that  $f_u = 0$  for  $u \in \mathbf{c} - X$ . If  $\Gamma \in L$ , then  $\hat{\mathbf{J}}_{\Gamma}$  is a  $\mathbf{J}_{\mathbf{c}}$ -submodule of  $\hat{\mathbf{J}}_{\mathbf{c}}$ . We have  $\hat{\mathbf{J}}_{\Gamma} = \bigoplus_{\Gamma' \in L} \hat{\mathbf{J}}_{\Gamma \cap \Gamma'^{-1}}$ .

According to a conjecture in [L4, 10.5], proved in [BFO], we can find  $G, X$  as in 4.1 and an isomorphism of  $\mathbf{C}$ -algebras  $\chi : \mathbf{K}_G(X \times X) \xrightarrow{\sim} \mathbf{J}_{\mathbf{c}}$  carrying the basis  $\{V^{\Omega, \rho}; (\Omega, \rho) \in B\}$  of  $\mathbf{K}_G(X \times X)$  onto the basis  $\{t_x; x \in \mathbf{c}\}$  of  $\mathbf{J}_{\mathbf{c}}$ . This extends in an obvious way to an isomorphism  $\hat{\chi} : \hat{\mathbf{K}}_G(X \times X) \xrightarrow{\sim} \hat{\mathbf{J}}_{\mathbf{c}}$  under which the left  $\mathbf{K}_G(X \times X)$ -module structure on  $\hat{\mathbf{K}}_G(X \times X)$  corresponds to the left  $\mathbf{J}_{\mathbf{c}}$ -module structure on  $\hat{\mathbf{J}}_{\mathbf{c}}$ . If  $\Gamma \in L$ , there is a unique  $\omega' \in G \setminus X$  (the set of  $G$ -orbits in  $X$ ) such that  $\hat{\chi}$  carries  $R_{\omega'}$  (see 4.3) onto a (simple)  $\mathbf{J}_{\mathbf{c}}$ -submodule  $M_{\Gamma}$  of  $\hat{\mathbf{J}}_{\Gamma}$  whose isomorphism class is independent of  $\Gamma$ ; we say that this is the special  $\mathbf{J}_{\mathbf{c}}$ -module. The  $\mathbf{J}_{\mathbf{c}}$ -module  $M_{\Gamma}$  admits a basis  $\{e_{\Gamma'}; \Gamma' \in L\}$  in which any  $t_u$  (with  $u \in \mathbf{c}$ ) acts by a matrix with all entries in  $\mathbf{Z}_{\geq 0}$ , namely the basis corresponding to the basis  $\{\tilde{V}^{\omega, \omega'}; \omega \in G \setminus X\}$  of  $R_{\omega'}$ .

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DEPARTMENT OF MATHEMATICS, M.I.T., CAMBRIDGE, MA 02139